EE 505 B Fall 2014 Assignment 1

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Problem 1

Using set algebra and the axioms of probability, show that:

$$Pr(A \cap \bar{B}) = Pr(A) - Pr(A \cap B)$$

Answer

$$\begin{aligned}
Pr(A \cap \bar{B}) &= Pr(A) - Pr(A \cap B) \\
Pr(A \cap \bar{B}) + Pr(A \cap B) &= Pr(A) \quad (1) \\
Pr(A \cap \bar{B} \cup A \cap B) &= Pr(A) \quad (2) \\
Pr(A \cap (\bar{B} \cup B)) &= Pr(A) \quad (3) \\
Pr(A \cap S) &= Pr(A) \quad (4) \\
Pr(A) &= Pr(A) \quad (5)
\end{aligned}$$

Following my steps above:

1. Simple Set Math applied to isolate $Pr(A)$.
2. Simple Set Algebra applied to combine $Pr(A \cap \bar{B})$ and $Pr(A \cap B)$.
3. Applied Distributive Property.
4. Applied Axiom 2.2 where $B \cup \bar{B} = S$.
5. Applied Product rule for $Pr(A \cap S) = Pr(A)$.

Alternatively, it seems to me that at Equation (1) we could apply the rule of total probability. Since $\bar{B}$ and B account for the total sample space S, the sum of the probabilities of A with each of them would necessarily describe the entire sample space of A. I have to admit a little confusion with using the axioms. I can see the answer using the rule of total probability but is it ok to apply the set algebra to combine the two probabilities? I am not certain.

Problem 2

For each of the following, provide justification for your answers.

(a) If sets A and B are mutually exclusive and collectively exhaustive, then are the sets $\bar{A}$ and $\bar{B}$ mutually exclusive?

(b) If sets A and B are mutually exclusive but not collectively exhaustive, then are the sets $\bar{A}$ and $\bar{B}$ mutually exclusive? Are they collectively exhaustive?

(c) If sets A and B are collectively exhaustive but not mutually exclusive, then are the sets $\bar{A}$ and $\bar{B}$ mutually exclusive? Are they collectively exhaustive?

Answer (a)

YES, sets $\bar{A}$ and $\bar{B}$ will be mutually exclusive.

$$\begin{aligned} A \cap B &= \varnothing \text{ mutually exclusive} \qquad (6) \\ A \cup B &= S \text{ collectively exhaustive} \qquad (7) \end{aligned}$$

If sets A and B are collectively exhaustive and mutually exclusive, then that means their opposite sets are equal to each other. With this being the case, a simple substitution on the mutually exclusive equation shows that $\bar{A}$ and $\bar{B}$ are indeed mutually exclusive:

$$\begin{aligned} A &= \bar{B} \\ B &= \bar{A} \\ \bar{B} \cap \bar{A} &= \varnothing, \text{ Commutative, } \bar{A} \cap \bar{B} = \varnothing \\ \bar{B} \cup \bar{B} &= S, \text{ Commutative, } \bar{A} \cap \bar{B} = S \end{aligned}$$

Answer (b)

NO, sets $\bar{A}$ and $\bar{B}$ will not be mutually exclusive.

$$\begin{aligned} A \cap B &= \varnothing \text{ mutually exclusive} \qquad (8) \\ A \cup B &\neq S \text{ not collectively exhaustive} \qquad (9) \end{aligned}$$

Since A and B are not collectively exhaustive, it is instructive to take a third set C which accounts for everything not in B or A so that $A \cup B \cup C = S$ and $C \cap A = \varnothing$ and $C \cap B = \varnothing$. With this in mind, $\bar{A}$ will contain all of C and $\bar{B}$ will also contain all of C meaning that A and B will definitely not be mutually exclusive.

YES, sets $\bar{A}$ and $\bar{B}$ will be collectively exhaustive.
Since $\bar{A}$ contains everything not in A including all of C and all of B and since $\bar{B}$ includes everything not in B including all of C and all of A, this makes $\bar{A}$ and $\bar{B}$ collectively exhaustive.

Answer (c)

$$A \cap B \neq \varnothing \text{ not mutually exclusive} \tag{10}$$
$$A \cup B = S \text{ collectively exhaustive} \tag{11}$$

YES, sets $\bar{A}$ and $\bar{B}$ will be mutually exclusive.
If $A \cap B \neq \varnothing$, then there is some overlap between A and B. This means that $\bar{A}$ and $\bar{B}$ will be:

$$\bar{A} = B - A \cap B$$
$$\bar{B} = A - A \cap B$$

Since both $\bar{A}$ and $\bar{B}$ will lack the only shared component of the space S, they will be mutually exclusive.

NO, sets $\bar{A}$ and $\bar{B}$ will not be collectively exhaustive.
As described for the mutually exclusive part of this question, $\bar{A}$ and $\bar{B}$ will each be missing a part of the total set S which means that their combined set cannot be collectively exhaustive.

Problem 3

In an experiment, you start at point A in Figure 1, and toss a coin. If the outcome of the toss is heads, you move on position to the right. If the outcome of the toss is tails, you move one poistion to the left. The experiment ends when you reach either point T_1 or point T_2. This is not a fair coin: the probability of heads is $\frac{2}{3}$.

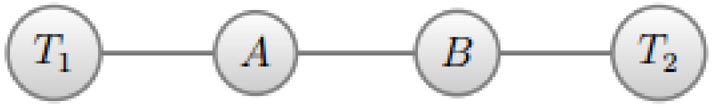

Figure 1: A random walk experiment.

(a) The sample space for this experiment is countably infinite. Write down the sample space for this experiment (as in Example 2.4 of the textbook). You obviously won't be able to write down *all* the elements in the sample space, so write down at least as many as you need to see a pattern.

(b) Write down the probabilities for six elementary events. From these probabilities, define a probability function for the number of tosses N until the experiment ends, i.e., find an expression (or expressions) for $Pr(N = n)$.

(c) Determine the probability that th experiment ends on or before the forth toss of the coin.

(d) Determine the probability that the experiment ends at point T_2. You can leave your answer in an unsimplified form.

(e) Determine the probability the event never ends.

Problem 4

(Problem 2.32 from the textbook) I deal myself 3 cards from a standard deck of 52 cards. Find the probabilities of each of the following events.

(a) 2 of a kind, e.g., 2 fives or 2 kings.

(b) 3 of a kind.

(c) 3 of the same suit, e.g., 3 hearts or 3 clubs.

(d) 3 cards in consecutive order, e.g., 2-3-4 or 10-J-Q.

Problem 5

The input-output relationship in a system is described by

$$y = 16e^{-1.6x}$$

where x is the input and y is the output. The input is in the range $-1 \leq x \leq 2$, and all values in this range occur in an equally likely fashion.

(a) Draw the sample spaces of the input and the output.

(b) Find these probabilities: $Pr[Y \leq 16]$ and $Pr[2 \leq Y < 20]$.

Problem 6

The transmission of signals over a trinary communications channel is represented in Figure 2, where the notation 0_T denotes the event "0 transmitted", 0_R denotes "0 received", etc.

Suppose that the input signals 0, 1, and 2 occur with probability $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{4}$, respectively.

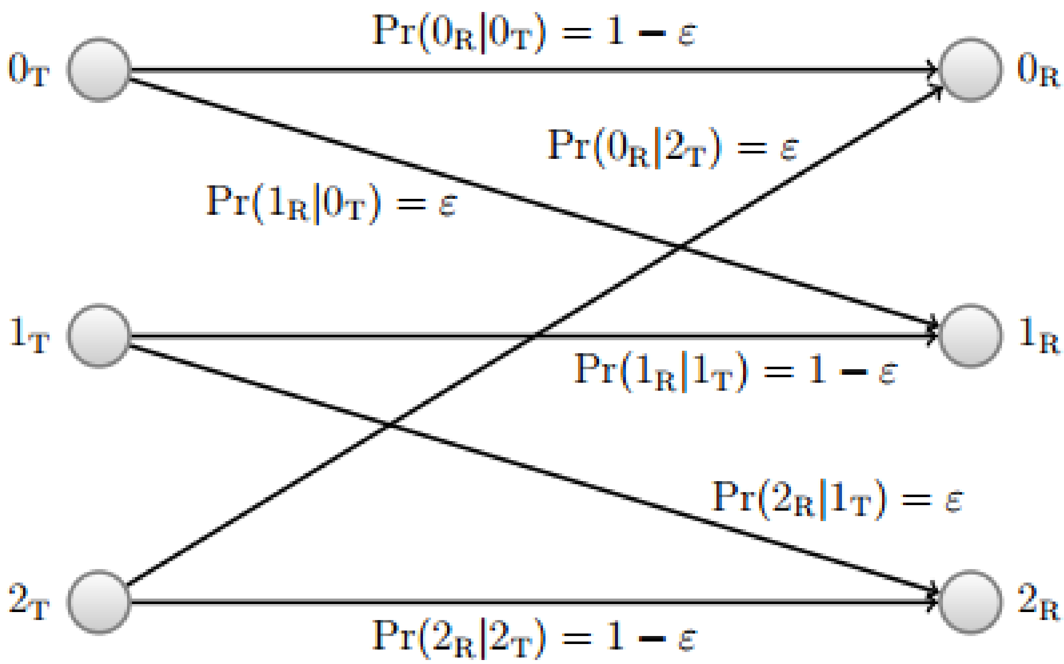

Figure 2: A representation of a binary communications channel.

(a) Find the probabilities of the output signals occurring, i.e., $Pr(0_R)$, $Pr(I_R)$, and $Pr(2_R)$.

(b) Find the probability of an error occurring.

(c) Find $Pr(0_T|1_R)$. If $\varepsilon = 1$, does your answer make sense? If $\varepsilon = 0$, does your answer make sense?

(d) Find $Pr(1_T|1_R)$. Again, do your answers make sense for $\varepsilon = 1$ and $\varepsilon = 0$?

(e) Find $Pr(2_T|1_R)$.

Problem 7

(MATLAB, optional) In this exercise, you are going to simulate the trinary channel in Problem 6. For this exercise, use a value of ε of 0.01.

(a) The following statement will generate a sequence of 10000 0s, 1s, and 2s where the probability of each occurring is $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{4}$ respectively.

```
>> N = 10000;
>> r = rand(1,N);
>> y = zeros(1, N);
>> x = (r>0.5 & r<=0.75)+2*(r>0.75);
```

To check that the sequences indeed have the desired relative frequency of 0s, 1s, and 2s, computer the relative frequency of occurrence of the 0s, 1s, and 2s in the input sequence.

(b) Simulate the channel by writing code to do the following:

- For a 0 in the input sequence, the code should generate an output of 0 with probability $Pr(0_R|0_T)$ and an output of 1 with probability $Pr(1_R|0_T)$.
- For a 1 in the input sequence, the code should generate an output of 1 with probability $Pr(1_R|1_T)$ and an output of 2 with probability $Pr(2_R|1_T)$.
- For a 2 in the input sequence, the code should generate an output of 2 with probability $Pr(2_R|2_T)$ and an output of 0 with probability $Pr(0_R|2_T)$.

(c) Use the input sequence generated in part (a) to estimate the probabilities $Pr(0_T|1_R)$, $Pr(0_T|1_R)$, and $Pr(error)$. To estimate the probability, use relative frequency; for example, to estimate $Pr(0_T|1_R)$ you would computer

$$\frac{\text{number of times 0 transmitted and 1 received}}{\text{number of times 1 received}}$$

(d) Compare the estimated values to the theoretical values computed in Problem 6. You may need to use a longer input sequence (e.g., 100000) to get stable results.