# Report: Multivariable Modeling of Electrostatic Actuator Dynamics

University of Washington
EE 557 Autumn 2015

Darrell Ross

October 27, 2015

# Contents

# 1 Abstract

In this problem, an analog implementation of positioning capability is developed for a simplified actuator that is part of a Digital Micromirror Device (DMD). A schematic showing the mirror (moveable electrode) is shown in Figure A.1.

The Basic Governing equations are shown in Appendix **A.1**. The Governing Differential Equations formed from them are shown in (1.1), (1.2), and (1.3).

$$\frac{dx}{dt} = v \tag{1.1}$$

$$\frac{dv}{dt} = \frac{1}{M_p}\left[\frac{1}{2}V_p^2\frac{\varepsilon_0\varepsilon_r A_c}{(x_0-x)^2} - K_s x - f_d - b_p v\right] \tag{1.2}$$

$$\frac{dV_p}{dt} = \frac{V_{in}-V_p}{R_{in}C_p(x)} = \frac{(V_{in}-V_p)(x_0-x)}{R_{in}\varepsilon_0\varepsilon_r A_c} \tag{1.3}$$

# 2 State Space Model

The Governing Differential Equations are non–linear and somewhat opaque. The State Space Block Diagram shown in Figure 2.1 helps us intuit the behavior. The first node covers the electrical portion of the system and is formed from Kirchoff's Voltage Law (KVL). The supply voltage $V_{in}(t)$ is equivalent to the sum of the voltage across the resistor $R_{in}$ and the plate voltage $V_p$, hence the positive feedback of $V_p$ shown in the diagram.

The second node covers the mechanical portion of the system and is formed using conservation of energy. The electrostatic force $f_e$ is the primary coupling point between the electrical energy of the voltage and the mechanical forces on the position of the plate, converting voltage to force per meter. The second node has feedback in terms of viscous damping $b_p v$ and the stiffness of the spring $K_s x$. The position of the plate $x$ also feeds back to the first equation effecting the parallel plate capacitance $C_p(x)$.

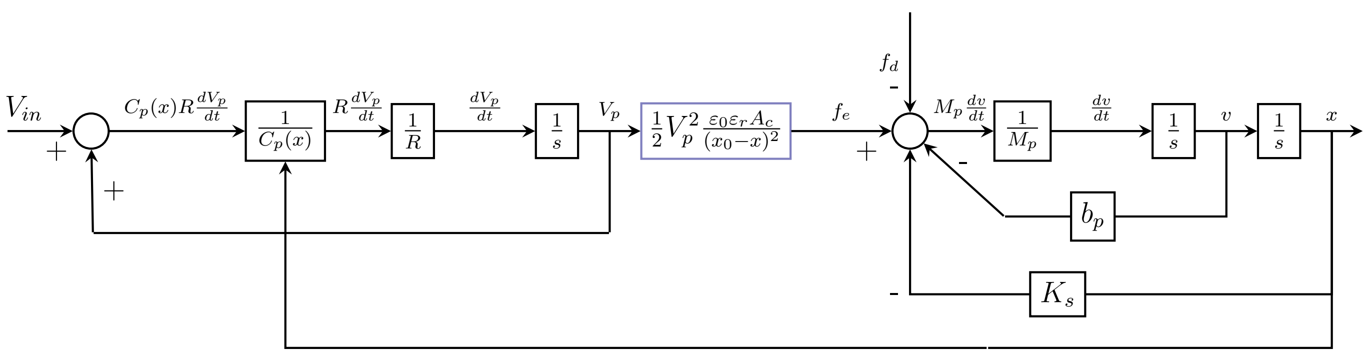


**Figure 2.1:** The state space model in block diagram form. It is important to note that the capacitance function is $C_p(x) = \frac{\varepsilon_0\varepsilon_r A_c}{x_0-x}$.

# 3 Operating Point Model

The Governing Differential Equations shown in (1.1), (1.2), and (1.3) are nonlinear and complicated. An Operating Point Model is created using Taylor Series Approximation with the results in (3.1), (3.2), and (3.3). The derivation of the Operating Point Model is covered in detail in Appendix B.

$$\frac{d\Delta x}{dt} = \Delta v \tag{3.1}$$

$$\frac{d\Delta v}{dt} = \left(\frac{V_{pop}^2 \varepsilon_0 \varepsilon_r A_c}{M_p(x_0 - x_{op})^3} - \frac{K_s}{M_p}\right)\Delta x - \left(\frac{b_p}{M_p}\right)\Delta v + \left(\frac{V_{pop}\varepsilon_0 \varepsilon_r A_C}{M_p(x_0 - x_{op})^2}\right)\Delta V_p - \left(\frac{1}{M_p}\right)\Delta f_d \tag{3.2}$$

$$\frac{d\Delta V_p}{dt} = \left(-\frac{V_{inop} - V_{pop}}{R_{in}\varepsilon_0 \varepsilon_r A_c}\right)\Delta x - \left(\frac{x_0 - x_{op}}{R_{in}\varepsilon_0 \varepsilon_r A_c}\right)\Delta V_p + \left(\frac{x_0 - x_{op}}{R_{in}\varepsilon_0 \varepsilon_r A_c}\right)\Delta V_{in} \tag{3.3}$$

The Operating Point equations can be reduced to (3.4), (3.5), and (3.6) using the following observations:

1. At the operating point where the system is steady–state, $V_{in} = V_p$ because there will be no current across the resistor $R_{in}$. This eliminates the $\Delta x$ term from (3.3).
2. The remaining two terms in (3.3) can be combined.
3. $\frac{1}{M_p}$ can be factored out of every entry in (3.2).

The Operating Point equations are shown in block diagram form in Figure 3.1.

$$\frac{d\Delta x}{dt} = \Delta v \tag{3.4}$$

$$\frac{d\Delta v}{dt} = \frac{1}{M_p}\left[\left(\frac{V_{pop}^2 \varepsilon_0 \varepsilon_r A_c}{(x_0 - x_{op})^3} - K_s\right)\Delta x - b_p \Delta v + \left(\frac{V_{pop}\varepsilon_0 \varepsilon_r A_C}{(x_0 - x_{op})^2}\right)\Delta V_p - \Delta f_d\right] \tag{3.5}$$

$$\frac{d\Delta V_p}{dt} = \frac{(\Delta V_{in} - \Delta V_p)}{R_{in}} \frac{(x_0 - x_{op})}{\varepsilon_0 \varepsilon_r A_c} \tag{3.6}$$

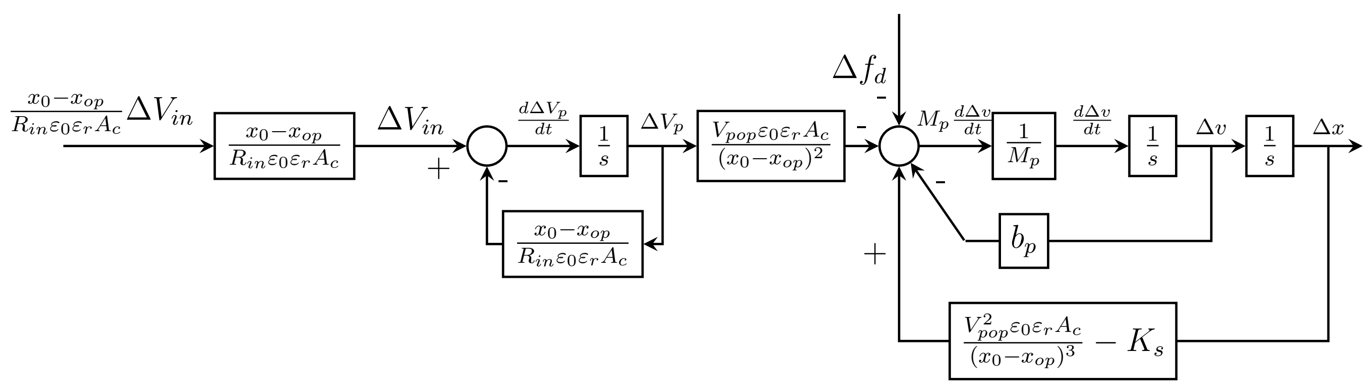


**Figure 3.1:** The Operating Point Model in block diagram form.

# 4 Reduced Operating Point Model

Beginning with (3.4), (3.5), and (3.6), the Operating Point Constants, $C_0$, $K_0$, and $F_0$ can be solved for by comparing to the provided Operating Point Equations from the assignment shown in (A.1), (A.2), (A.3). Detailed derivations for each constant are shown in Appendix C.

### 4.1 Operating Point Capacitance, $C_0$

$$C_0 = \frac{\varepsilon_0 \varepsilon_r A_c}{x_0 - x_{op}} = C_p(x_{op}) \tag{4.1}$$

### 4.2 Operating Point Stiffness, $K_0$

$$K_0 = \frac{V_{pop}^2 \varepsilon_0 \varepsilon_r A_c}{(x_0 - x_{op})^3} \tag{4.2}$$

### 4.3 Operating Point Charge per Meter, $F_0$

$$F_0 = \frac{V_{pop} \varepsilon_0 \varepsilon_r A_c}{(x_0 - x_{op})^2} \tag{4.3}$$

### 4.4 Reduced Block Diagram

Using the reduced Operating Point Model, the block diagram can be greatly simplified as shown in Figure 4.2.

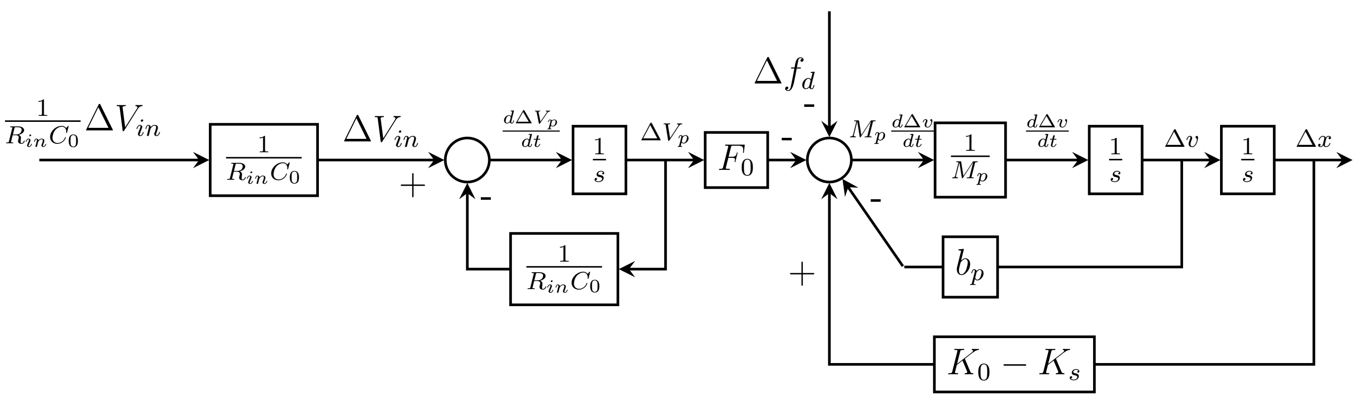


**Figure 4.2:** The Reduced Operating Point Model in block diagram form.

# 5 Transfer Function

The transfer function is found by converting the simplified version of the Operating Point Equations ((A.1), (A.2), (A.3)) to the frequency domain via a Laplace Transform. It is convenient to separate the $\Delta V_p$ and $\Delta V_{in}$ terms similar to (3.3) since, ultimately, we only care about $\Delta f_d$ which means we set $\Delta V_{in} = 0$.

This process is completed using the constants $F_0$, $C_0$, and $K_0$ calculated earlier.

$$s\Delta x = \Delta v \tag{5.1}$$

$$s\Delta v = \frac{1}{M_p}\left[(K_0 - K_s)\Delta x - b_p \Delta v - F_0 \Delta V_p - \Delta f_d\right] \tag{5.2}$$

$$s\Delta V_p = \frac{1}{R_{in}C_0}\Delta V_{in} - \frac{1}{R_{in}C_0}\Delta V_p \tag{5.3}$$

# 6 Dynamic Stiffness, $\frac{f_d}{x}$

Solving for Dynamic Stiffness is shown in Appendix E. The result is shown here in (6.1).

$$\therefore \frac{\Delta f_d}{\Delta x} = K_0 - K_s - b_p s - M_p s^2 \tag{6.1}$$

*where*

$$K_0 = \frac{V_{pop}^2 \varepsilon_0 \varepsilon_r A_c}{(x_0 - x_{op})^3}$$

$$V_{pop} = \pm\sqrt{\frac{2K_s x_{op}(x_0 - x_{op})^2}{\varepsilon_0 \varepsilon_r A_c}}$$

## 6.1 Dynamic Stiffness Plot

The dynamic stiffness function is shown plotted in Figure 6.1. In the plot of the frequency response function, there is a clear resonant frequency when $x_{op} = 0.01mm$ and $x_{op} = 0.03mm$. The frequencies are $4.43Hz$ and $1.90Hz$ respectively. The remaining larger values of $x_{op}$ lack resonant frequencies.

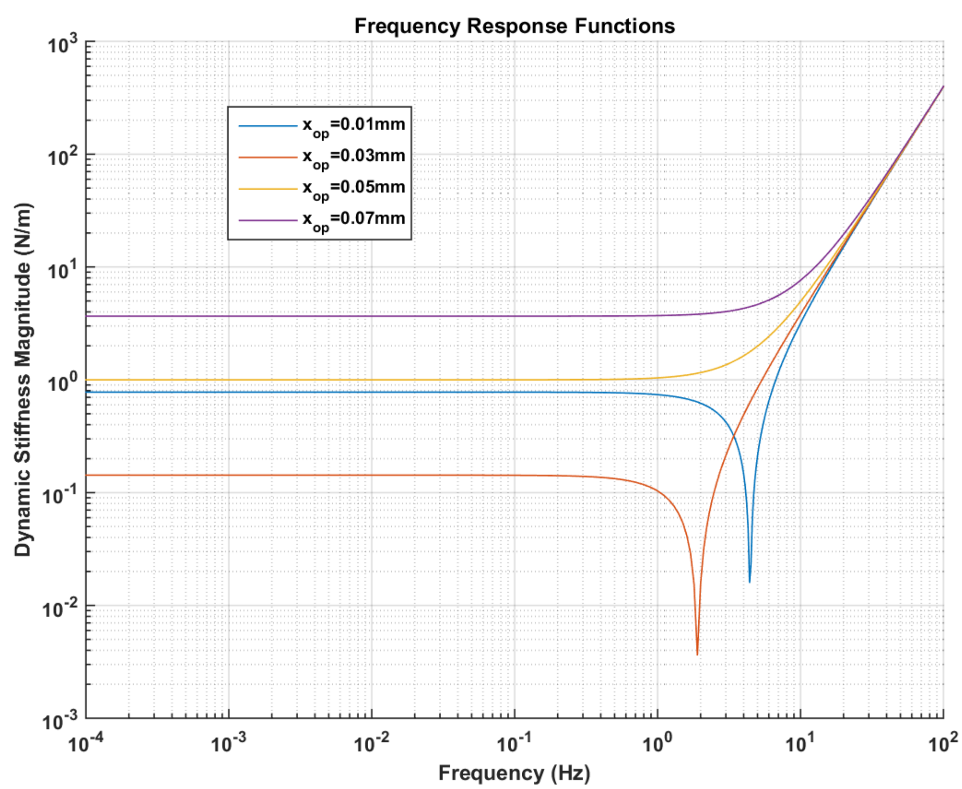


**Figure 6.1:** A plot of Dynamic Stiffness vs Frequency for four values of $x_{op}$.

Resonant frequencies here are bad as they represent holes in the dynamic stiffness. A further analysis of Resonant Frequencies is conducted in Appendix F.

## 6.2 Unit Verification

Verifying that all units work out is very important. For this process, the $K_0$ term was produced by substituting $V_{pop}$ into the equation for $K_0$. First, note that the reduced term for $K_0$ has a $[mm]$ value divided by a $[mm]$ value. This means the units cancel which means that correcting for meters instead of millimeters is unnecessary. The $M_p s^2$ term is in grams and needs to be in killograms. This is corrected for by modifying the $M_p$ input to be 0.001 instead of 1.0.

$$\begin{aligned}
\frac{\Delta f_d}{\Delta x} &= \frac{2K_s x_{op}}{(x_0 - x_{op})} - K_s - b_p s - M_p s^2 \\
\frac{[N]}{[mm]} &= \frac{[N/m]\,[mm]}{[mm]} - [N/m] - \frac{[N]}{[m/s]}\frac{[1]}{[s]} - \frac{[g]}{[s^2]} \\
\frac{[N]}{[mm]} &= \frac{[N]}{[m]} - \frac{[N]}{[m]} - \frac{[N]}{[m]} - \frac{[g]}{[s^2]}
\end{aligned} \tag{6.2}$$

Although it at first appeared that the last term of (6.2) was not correct, closer inspection, shown in (6.3) and (6.4), shows that all units check out.

$$[N] = \frac{[kg]\,[m]}{[s^2]} \tag{6.3}$$

$$\frac{[N]}{[m]} = \frac{[kg]}{[s^2]} \tag{6.4}$$

All units reduce to N/m which makes sense since we are relating the amount of disturbance force $f_d$ required to change the location of the plate $x$.

# 7 Conclusions

This report covers the formulation of the transfer functions and analysis of dynamic stiffness for the DMD. For the operating point model, there is a resonant frequency when $x_{op} \leq 0.03mm$. When $x_{op}$ is greater than 0.03mm, the resonant frequency goes away. This suggests that in the design of this device, the plate distance operating point should move more than 0.03mm. It would be acceptable to move less than 0.03mm but only if the resonant frequency range for disturbance force $f_d$ could be guaranteed to be avoided. With the resonance in the range of $2Hz$ to $6Hz$ it seems like avoiding them all together would be a better bet.

If this were a digital function instead of analog. That is, if the plate had only two positions, $x_0$ and $x_{op}$, then it should be designed such that $x_{op} > 0.033333mm$. Ideally, it should be a bit more than that so as to avoid the effects of the trough, shown in Figure F.1, entirely.

The Dynamic Stiffness plot can be shifted left by increasing $M_p$ and right by decreasing it.

# A Appendix: Problem Description

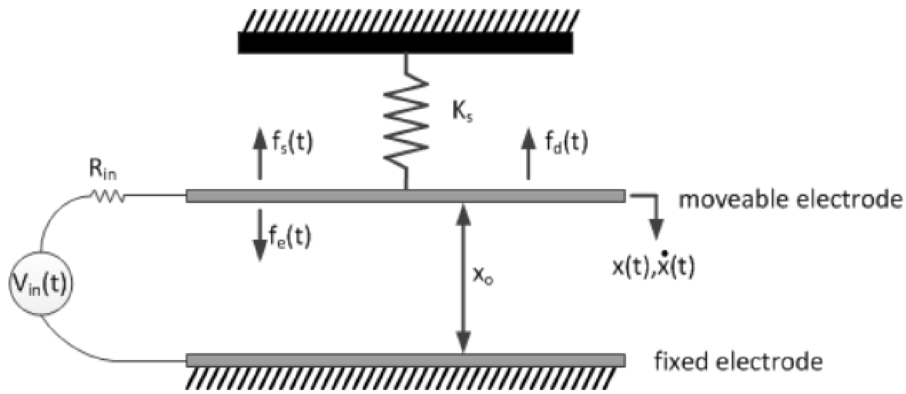


**Figure A.1:** A schematic diagram of the Digital Micromirror Device (DMD).

Variables of the DMD are defined as:

$$
\begin{aligned}
C_p(x) &= \text{parallel plate capacitance} \\
R &= \text{source resistance} = 1.0 \text{ ohm} \\
\varepsilon_0 &= \text{permittivity of empty space} = 4\pi 10^{-7}\frac{C^2}{N-m^2} \\
\varepsilon_r &= \text{relative permittivity} = 1\text{for air} \\
x_0 &= \text{initial open circuit position} = 0.1\text{mm} \\
f_e &= \text{electrostatic air gap force} \\
f_d &= \text{external load disturbance force} \\
x &= \text{position of moving plate w.r.t. fixed plate} \\
A_c &= \text{capacitor plate cross sectional area} = 1\text{in}^2 \\
b_p &= \text{viscous damping} = 3\times 10^{-4}\frac{N}{m/s} \\
M_p &= \text{mass of moving plate} = 1\text{gm} \\
K_s &= \text{stiffness of spring} = 1N/m
\end{aligned}
$$

## A.1 Governing Equations

The basic governing equations for the DMD shown in Figure A.1 are shown here.

$$C_p(x) = \frac{\varepsilon_0\varepsilon_r A_c}{x_0 - x} \qquad \text{(capacitance)}$$

$$f_e = \frac{1}{2}V_p^2\frac{\varepsilon_0\varepsilon_r A_c}{(x_0-x)^2} \qquad \text{(electrostatic force)}$$

$$f_x = K_s x \qquad \text{(spring force)}$$

$$V_{in} = iR + V_p, \text{where } i = C_p(x)\frac{dV_p}{dt} \qquad \text{(voltage loop)}$$

$$M_p\frac{dv}{dt} = f_e - f_s - f_d - b_p v \qquad \text{(Newton's Law)}$$

## A.2 Operating Point Equations

The operating point equations from the assignment are shown in (A.1), (A.2), and (A.3).

$$\frac{d\Delta x}{dt} = \Delta v \tag{A.1}$$

$$\frac{d\Delta v}{dt} = \frac{1}{M_p}[(K_0 - K_s)\Delta x - F_0\Delta V_p - \Delta f_d - b_p\Delta v] \tag{A.2}$$

$$\frac{d\Delta V_p}{dt} = \frac{\Delta V_{in} - \Delta V_p}{R_{in}C_0} \tag{A.3}$$

# B Appendix: Transfer Function Methods

Several methods exist for finding the Transfer Functions.

## B.1 Direct Substitution

If the equations are simple enough, this solution is possible. Given the nature of this course, it is unlikely we will get to apply this method.

1. Convert the time domain equations to the frequency domain using a LaPlace Transform.
2. Use direct substitution to solve for the desired variables for each Transfer Function relating each input to each output.

## B.2 State Space Approach

For this method, the equations are placed in matrix form and matrix math can be used to solve all the transfer functions at the same time.

1. The equations are first stated in vector form as shown in (B.1).
2. The LaPlace Transform is done on each entry in the matrices.
3. Some matrix math is used to solve for the function of the state vector in one shot.
4. Finally the Transfer Function Matrix is produced.

$$\begin{bmatrix} \dot{x} \\ \dot{v} \\ \dot{V}_p \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial V_p} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial v} & \frac{\partial v}{\partial V_p} \\ \frac{\partial V_p}{\partial x} & \frac{\partial V_p}{\partial v} & \frac{\partial V_p}{\partial V_p} \end{bmatrix} \begin{bmatrix} x \\ v \\ V_p \end{bmatrix} + \begin{bmatrix} \frac{\partial x}{\partial V_{in}} & \frac{\partial x}{\partial f_d} \\ \frac{\partial v}{\partial V_{in}} & \frac{\partial v}{\partial f_d} \\ \frac{\partial V_p}{\partial V_{in}} & \frac{\partial V_p}{\partial f_d} \end{bmatrix} \begin{bmatrix} V_{in} \\ f_d \end{bmatrix} \tag{B.1}$$

$$sX(s) = AX(s) + BU(s)$$

$$X(s) = [sI - A]^{-1}BU(s)$$

where

$$[sI - A]^{-1} = \frac{1}{Det[sI - A]}Adj[sI - A]$$

The full transfer function matrix for this problem is shown in (D.2), although it was solved using the next method.

## B.3 Operating Point Model Method

When the system of equations is complicated, converting to the frequency domain with the LaPlace Transform can be very complicated. In these situations, the Operating Point Model can be used.

1. The system is solved at an Operating Point using Taylor Series Approximation (this is often called linearization).
2. Once a linear set of equations is produced, they are easily converted to the frequency domain via a LaPlace Transform.
3. The resulting frequency domain functions are used to solve for the transfer functions using direct substitution.

# C Appendix: Operating Point Model Solution

Following the steps outlined in Appendix **B.3**, the Operating Point Model is derived here.

## C.1 Taylor Series Approximation

The Taylor Series Approximation is a series expansion of a function about an operating point. The delta–values represent a figurative change needed to determine the secant line at the operating point.

$$\begin{aligned}
&\mathcal{F}(x_{op}+\Delta x, v_{op}+\Delta v, V_{pop}+\Delta V_p, V_{inop}+\Delta V_{in}, f_{dop}+\Delta f_d) = \\
&\mathcal{F}(x_{op}, v_{op}, V_{pop}, V_{inop}, f_{dop}) + \Delta x \left.\frac{\partial \mathcal{F}}{\partial x}\right|_{op} + \Delta v \left.\frac{\partial \mathcal{F}}{\partial v}\right|_{op} + \Delta V_p \left.\frac{\partial \mathcal{F}}{\partial V_p}\right|_{op} + \Delta V_{in} \left.\frac{\partial \mathcal{F}}{\partial V_{in}}\right|_{op} + \Delta f_d \left.\frac{\partial \mathcal{F}}{\partial f_d}\right|_{op} + \\
&\Delta x^2 \left.\frac{\partial^2 \mathcal{F}}{\partial x^2}\right|_{op} + \Delta v^2 \left.\frac{\partial^2 \mathcal{F}}{\partial v^2}\right|_{op} + \Delta V_p^2 \left.\frac{\partial^2 \mathcal{F}}{\partial V_p^2}\right|_{op} + \Delta V_{in}^2 \left.\frac{\partial^2 \mathcal{F}}{\partial V_{in}^2}\right|_{op} + \Delta f_d^2 \left.\frac{\partial^2 \mathcal{F}}{\partial f_d^2}\right|_{op} + \\
&\Delta x^3 \left.\frac{\partial^3 \mathcal{F}}{\partial x^3}\right|_{op} + \Delta v^3 \left.\frac{\partial^3 \mathcal{F}}{\partial v^3}\right|_{op} + \Delta V_p^3 \left.\frac{\partial^3 \mathcal{F}}{\partial V_p^3}\right|_{op} + \Delta V_{in}^3 \left.\frac{\partial^3 \mathcal{F}}{\partial V_{in}^3}\right|_{op} + \Delta f_d^3 \left.\frac{\partial^3 \mathcal{F}}{\partial f_d^3}\right|_{op} + \ldots
\end{aligned}$$

$$\begin{aligned}
\therefore \Delta \mathcal{F} &= \mathcal{F}(x_{op}+\Delta x, v_{op}+\Delta v, V_{pop}+\Delta V_p, V_{inop}+\Delta V_{in}, f_{dop}+\Delta f_d) - \mathcal{F}(x_{op}, v_{op}, V_{pop}, V_{inop}, f_{dop}) \\
&= \Delta x \left.\frac{\partial \mathcal{F}}{\partial x}\right|_{op} + \Delta v \left.\frac{\partial \mathcal{F}}{\partial v}\right|_{op} + \Delta V_p \left.\frac{\partial \mathcal{F}}{\partial V_p}\right|_{op} + \Delta V_{in} \left.\frac{\partial \mathcal{F}}{\partial V_{in}}\right|_{op} + \Delta f_d \left.\frac{\partial \mathcal{F}}{\partial f_d}\right|_{op}
\end{aligned}$$

The partial derivative of each function of the Governing Differential Equations with respect to each input and output variable produces the A–matrix and B–matrix shown in (C.1). These equations are shown in Block Diagram Form in Figure 3.1 and in equation form in (3.1), (3.2), and (3.3).

$$\begin{bmatrix} \frac{d\Delta x}{dt} \\ \frac{d\Delta v}{dt} \\ \frac{d\Delta V_p}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{V_{pop}^2 \varepsilon_0 \varepsilon_r A_c}{M_p (x_0 - x_{op})^3} - \frac{K_s}{M_p} & -\frac{b_p}{M_p} & \frac{V_{pop} \varepsilon_0 \varepsilon_r A_c}{M_p (x_0 - x_{op})^2} \\ -\frac{V_{inop} - V_{pop}}{R_{in} \varepsilon_0 \varepsilon_r A_c} & 0 & -\frac{x_0 - x_{op}}{R_{in} \varepsilon_0 \varepsilon_r A_c} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta v \\ \Delta V_p \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{M_p} \\ \frac{x_0 - x_{op}}{R_{in} \varepsilon_0 \varepsilon_r A_c} & 0 \end{bmatrix} \begin{bmatrix} \Delta V_{in} \\ \Delta f_d \end{bmatrix} \tag{C.1}$$

## C.2 Solving For Constants

Comparing the reduced Taylor Series Approximation equations ((3.4), (3.5), (3.6)) with the provided Operating Point Equations ((A.1), (A.2), (A.3)), $K_0$, $F_0$, and $C_0$ can be solved for.

### Operating Point Capacitance, $C_0$

The Operating Point Capacitance in (A.3), $C_0$, is easily the inverse of the remaining piece of the equation shown in (3.6) which is also equivalent to the Capacitance equation at the operating point $x_{op}$.

$$\frac{\Delta V_{in} - \Delta V_p}{R_{in}C_0} = \frac{\Delta V_{in} - \Delta V_p}{R_{in}C_0}\frac{\varepsilon_0\varepsilon_r A_c}{x_0 - x}$$

$$C_0 = \frac{\varepsilon_0\varepsilon_r A_c}{x_0 - x_{op}} = C_p(x_{op}) \tag{C.2}$$

### Operating Point Stiffness, $K_0$

Since $K_0$ is shown along with $K_s$ in the product with $\Delta x$ in (A.2), the Operating Point Stiffness in (A.2), $K_0$, is easily seen when reviewing (3.5). Operating Point Stiffness is a function of the plate voltage $V_{pop}$ and plate location $x_{op}$ at the operating point.

$$\frac{1}{M_p}\left[\left(\frac{V_{pop}^2\varepsilon_0\varepsilon_r A_c}{(x_0 - x_{op})^3} - K_s\right)\right]\Delta x = \frac{1}{M_p}(K_0 - K_s)\Delta x$$

$$\left(\frac{V_{pop}^2\varepsilon_0\varepsilon_r A_c}{(x_0 - x_{op})^3} - K_s\right)\Delta x = (K_0 - K_s)\Delta x$$

$$\therefore K_0 = \frac{V_{pop}^2\varepsilon_0\varepsilon_r A_c}{(x_0 - x_{op})^3} \tag{C.3}$$

### Operating Point Charge per Meter, $F_0$

The Operating Point Charge per Meter in (A.2), $F_0$, is easily seen when reviewing (3.5). Since $F_0$ is multiplied only by $\Delta V_p$, the term can be reduced. Similar to Operating Point Stiffness, Operating Point Charge per Meter is a function of plate voltage $V_{pop}$ and plate distance location $x_{op}$ at the operating point.

$$\frac{V_{pop}\varepsilon_0\varepsilon_r A_c}{(x_0 - x_{op})^2}\Delta V_p = F_0\Delta V_p$$

$$\therefore F_0 = \frac{V_{pop}\varepsilon_0\varepsilon_r A_c}{(x_0 - x_{op})^2} \tag{C.4}$$

# D Full Transfer Function Matrix

Out of curiosity, I solved the entire transfer function matrix using substitution and doing one at a time. The final result is shown in (D.2).

$$\begin{bmatrix} x(s) \\ v(s) \\ V_p(s) \end{bmatrix} = \begin{bmatrix} H_{xV_{in}}(s) & H_{xf_d}(s) \\ H_{vV_{in}}(s) & H_{vf_d}(s) \\ H_{V_pV_{in}}(s) & H_{V_pf_d}(s) \end{bmatrix} \begin{bmatrix} V_{in}(s) \\ f_d(s) \end{bmatrix} \tag{D.1}$$

$$\begin{bmatrix} x(s) \\ v(s) \\ V_p(s) \end{bmatrix} = \frac{1}{(K_0 - K_s - b_p s - M_p s^2)} \begin{bmatrix} \frac{F_0}{(1+R_{in}C_o s)} & 1 \\ \frac{F_0 s}{(1+R_{in}C_o s)} & 0 \\ \frac{1}{(1+R_{in}C_o s)} & s \end{bmatrix} \begin{bmatrix} V_{in}(s) \\ f_d(s) \end{bmatrix} \tag{D.2}$$

Multiplying the results out from (D.2), we get (D.3), (D.4), and (D.5).

$$\Delta x = \frac{F_0}{(K_0 - K_s - b_p s - M_p s^2)(1 + R_{in} C_0 s)} \Delta V_{in} + \frac{1}{(K_0 - K_s - b_p s - M_p s^2)} \Delta f_d \tag{D.3}$$

$$\Delta v = \frac{F_0 s}{(K_0 - K_s - b_p s - M_p s^2)(1 + R_{in} C_0 s)} \Delta V_{in} + 0 \Delta f_d \tag{D.4}$$

$$\Delta x = \frac{1}{(K_0 - K_s - b_p s - M_p s^2)(1 + R_{in} C_0 s)} \Delta V_{in} + \frac{s}{(K_0 - K_s - b_p s - M_p s^2)} \Delta f_d \tag{D.5}$$

With enough time and patience, a complete sensitivity analysis could be done using these functions.

# E Appendix: Dynamic Stiffness Derivation

Dynamic Stiffness is solved using superposition to look at $\Delta f_d$, setting $\Delta V_{in} = 0$. Looking at (5.3), the only solution for $\Delta V_p$ is $\Delta V_p = 0$. We also know from (5.1) that $\Delta v = s\Delta x$. Making these two substitutions into (5.2) and doing some basic algebraic manipulation, we can solve for the Dynamic Stiffness shown in (E.1).

$$\begin{aligned}
s(s\Delta x) &= \frac{1}{M_p} \left[ (K_0 - K_s)\Delta x - b_p(s\Delta x) - \Delta f_d \right] \\
\Delta f_d &= (K_0 - K_s)\Delta x - b_p s \Delta x - M_p s^2 \Delta x \\
\therefore \frac{\Delta f_d}{\Delta x} &= K_0 - K_s - b_p s - M_p s^2 \qquad \text{(E.1)} \\
\textit{where} & \\
K_0 &= \frac{V_{pop}^2 \varepsilon_0 \varepsilon_r A_c}{(x_0 - x_{op})^3}
\end{aligned}$$

The only unknown variable in (E.1) is $V_{pop}$ which can be solved for using the original Governing Differential Equation for $\frac{dv}{dt}$, (1.2).

At the operating point in steady state, velocity $v_{op}$ and acceleration $\frac{dv}{dt}$ are zero. The disturbance force $f_{dop}$ is also zero. This greatly reduces the equation making it possible to solve for $V_{pop}$.

$$\begin{aligned}
\frac{dv}{dt} &= \frac{1}{M_p} \left[ \frac{1}{2} V_{pop}^2 \frac{\varepsilon_0 \varepsilon_r A_c}{(x_0 - x_{op})^2} - K_s x_{op} - f_{dop} - b_p v_{op} \right] \\
0 &= \frac{1}{M_p} \left[ \frac{1}{2} V_{pop}^2 \frac{\varepsilon_0 \varepsilon_r A_c}{(x_0 - x_{op})^2} - K_s x_{op} \right] \\
\frac{1}{2} V_{pop}^2 \frac{\varepsilon_0 \varepsilon_r A_c}{(x_0 - x_{op})^2} &= K_s x_{op} \\
\therefore V_{pop} &= \pm \sqrt{\frac{2 K_s x_{op} (x_0 - x_{op})^2}{\varepsilon_0 \varepsilon_r A_c}} \qquad \text{(E.2)}
\end{aligned}$$

# F Appendix: Resonant Frequency Analysis

The resonant frequency could be a big problem so a closer look was warranted. To get a clearer picture of where the resonant frequency is across a range of $x_{op}$ values, the three-dimensional plot in Figure F.1 was produced.

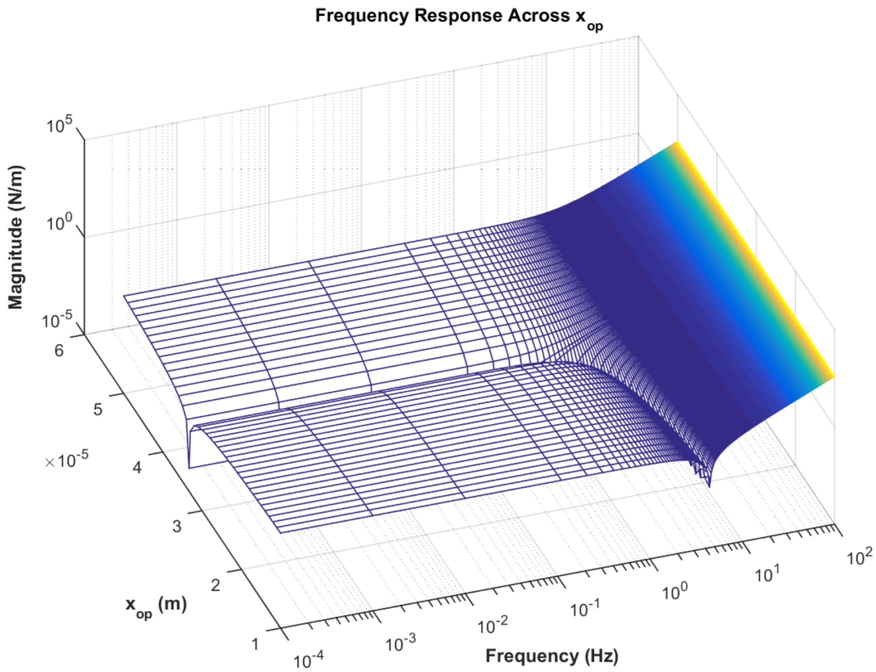


**Figure F.1:** A plot showing Frequency ($f$) vs Plate Position ($x_{op}$) vs Magnitude ($\frac{\Delta f_d}{\Delta x}$) shows a better big picture of what is going on with the resonance frequency.

In Figure F.1, there is a clear trough that designers of the DMD should stay away from if at all possible. Digging a little bit further, Figure F.3 shows the frequency response moving away from the trough. It is clear that using an $x_{op}$ value larger than 0.033333mm is desirable.

Solving the Transfer Function at the trough location happens when $x_{op} = \frac{1}{30}mm$ or $0.033\bar{3}mm$ and the frequency $f$ is 0.0001Hz yields a Dynamic Stiffness of 0.00014999. I used this value to solve the function for $x_{op}$ in order to effectively create a contour plot of Figure F.1 within the trough. The result is shown in Figure F.2.

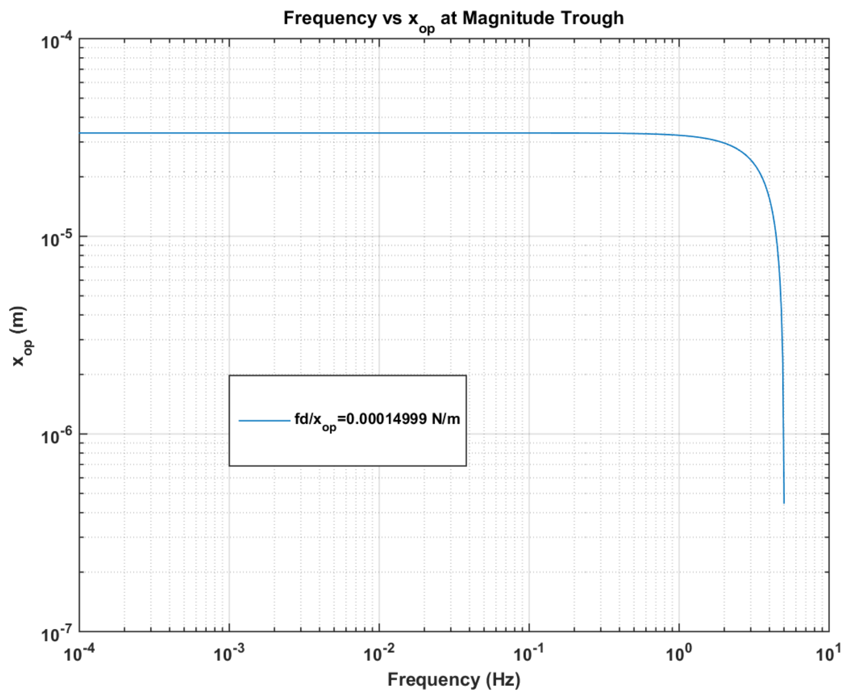


**Figure F.2:** A plot of Frequency vs Plate Position with the Dynamic Stiffness set to 0.00014999 N/m.

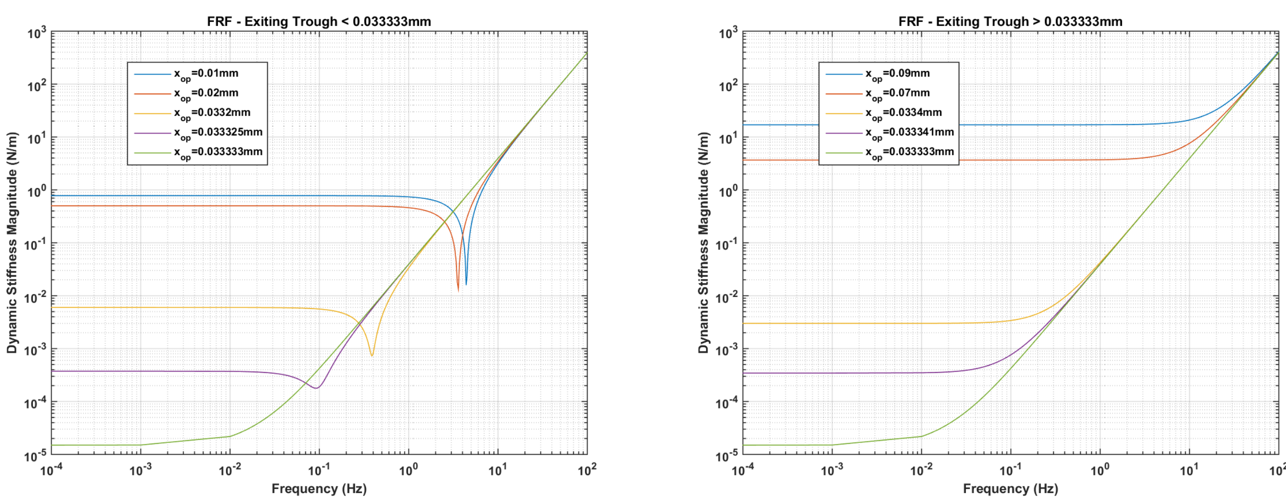


**Figure F.3:** As the operating point position $x_{op}$ is varied near the trough, getting smaller than 0.033333mm (left) leads to resonant frequencies while getting larger than 0.033333mm (right) leads to no resonant frequencies.